

EEE2035F: Signals and Systems I

Class Test 2

16 April 2010

SOLUTIONS

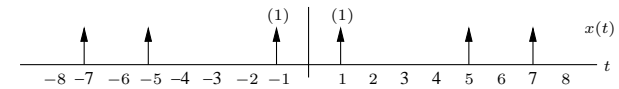
Name:

Student number:

Information

- The test is closed-book.
 - This test has *four* questions, totalling 20 marks.
 - Answer *all* the questions.
 - You have 45 minutes.
-

1. (5 marks) Find the complex exponential Fourier series representation of the periodic signal below:



Signal has period $T = 6$ so $\omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$ rad/s in an expansion

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

The Fourier coefficients are obtained by integrating over one period:

$$\begin{aligned} c_k &= \frac{1}{6} \int_{-6/2}^{6/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{6} \int_{-3}^3 [\delta(t+1) + \delta(t-1)] e^{-jk\omega_0 t} dt \\ &= \frac{1}{6} \int_{-3}^3 \delta(t+1) e^{-jk\omega_0 t} dt + \frac{1}{6} \int_{-3}^3 \delta(t-1) e^{-jk\omega_0 t} dt \\ &= \frac{1}{6} e^{-jk\omega_0(-1)} \int_{-3}^3 \delta(t+1) dt + \frac{1}{6} e^{-jk\omega_0(1)} \int_{-3}^3 \delta(t-1) dt \\ &= \frac{1}{6} e^{jk\omega_0} + \frac{1}{6} e^{-jk\omega_0}, \end{aligned}$$

This completely specifies the coefficients, although it would be easier to simplify to

$$c_k = \frac{2}{6} \frac{1}{2} (e^{jk\omega_0} + e^{-jk\omega_0}) = \frac{1}{3} \cos(k\omega_0)$$

if you had to calculate them.

2. (5 marks) Use transform properties to find the Fourier transform of

$$x(t) = e^{j5t} e^{-2(t-2)} u(t-2).$$

This can be decomposed as a time shift followed by multiplication by e^{j5t} . Start with

$$e^{-2t} u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2 + j\omega}$$

and apply time-shifting to get

$$e^{-2(t-2)} u(t-2) \xleftrightarrow{\mathcal{F}} \frac{1}{2 + j\omega} e^{-j\omega 2}$$

A frequency shift of 5 rad/s then yields the pair

$$e^{j5t} e^{-2(t-2)} u(t-2) \xleftrightarrow{\mathcal{F}} \frac{1}{2 + j(\omega - 5)} e^{-j2(\omega - 5)},$$

so the required Fourier transform is

$$X(\omega) = \frac{1}{2 + j(\omega - 5)} e^{-j2(\omega - 5)}.$$

3. (5 marks) Consider a signal with the following frequency domain representation:

$$X(\omega) = 2\pi[\delta(\omega - 5) + \delta(\omega + 5)].$$

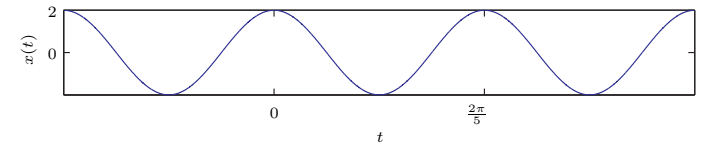
(a) Use the Fourier synthesis integral directly to show that $x(t) = 2 \cos(5t)$.

(b) Plot $x(t)$ and $X(\omega)$.

(a) The transform is

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi[\delta(\omega - 5) + \delta(\omega + 5)] e^{j\omega t} d\omega \\ &= e^{j5t} \int_{-\infty}^{\infty} \delta(\omega - 5) d\omega + e^{-j5t} \int_{-\infty}^{\infty} \delta(\omega + 5) d\omega \\ &= \frac{2}{2} (e^{j5t} + e^{-j5t}) = 2 \cos(5t). \end{aligned}$$

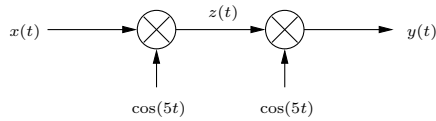
(b) Plots are



and



4. (5 marks) Consider the system shown below:



If the Fourier transform of $x(t)$ is $X(\omega) = p_1(\omega)$, then sketch the Fourier transform of $Y(\omega)$.

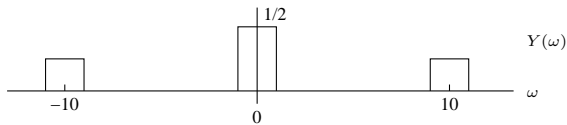
The transform of $z(t)$ is

$$Z(\omega) = \frac{1}{2}(X(\omega + 5) + X(\omega - 5)),$$

so

$$\begin{aligned} Y(\omega) &= \frac{1}{2}(Z(\omega + 5) + Z(\omega - 5)) \\ &= \frac{1}{4}[X((\omega + 5) + 5) + X((\omega + 5) - 5)] + \frac{1}{4}[X((\omega - 5) + 5) + X((\omega - 5) - 5)] \\ &= \frac{1}{4}X(\omega + 10) + \frac{1}{2}X(\omega) + \frac{1}{4}X(\omega - 10). \end{aligned}$$

For $X(\omega) = p_1(\omega)$ this looks like



INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t - c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{ a }X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \dots$
Frequency shift	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0 \text{ real}$
Multiplication by $\cos(\omega_0 t)$	$x(t) \cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \dots$
Integration	$\int_{-\infty}^t x(\lambda) d\lambda \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0) \delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega) d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) ^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
1 $(-\infty < t < \infty)$	$2\pi \delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi \delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t - c)$	$e^{-j\omega c} \quad (c \text{ any real number})$
$e^{-bt} u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi \delta(\omega - \omega_0) \quad (\omega_0 \text{ any real number})$
$p_\tau(t)$	$\tau \text{sinc} \frac{\omega\tau}{2}$
$\tau \text{sinc} \frac{\omega\tau}{2}$	$2\pi p_\tau(\omega)$
$(1 - \frac{2 t }{\tau}) p_\tau(t)$	$\frac{\pi}{2} \text{sinc}^2(\frac{\omega\tau}{4\pi})$
$\frac{\pi}{2} \text{sinc}^2 \frac{\omega\tau}{4\pi}$	$2\pi (1 - \frac{2 \omega }{\tau}) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta} \delta(\omega + \omega_0) + e^{j\theta} \delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta} \delta(\omega + \omega_0) - e^{j\theta} \delta(\omega - \omega_0)]$

Trigonometric identities

$$\begin{aligned} \sin(-\theta) &= -\sin(\theta) & \cos(-\theta) &= \cos(\theta) & \tan(-\theta) &= -\tan(\theta) \\ \sin^2(\theta) + \cos^2(\theta) &= 1 & \sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\ \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) \\ \sin(\theta_1 + \theta_2) &= \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & \cos(\theta_1 + \theta_2) &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\ e^{j\theta} &= \cos(\theta) + j\sin(\theta) \end{aligned}$$