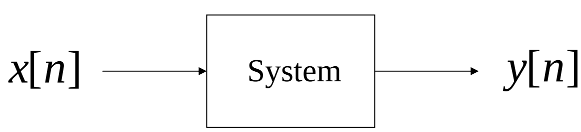
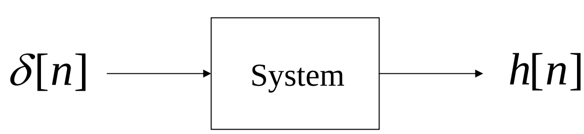
# Chapter 3
# Convolution Representation

## DT Unit-Impulse Response

- Consider the DT SISO system:




- If the input signal is $x[n] = \delta[n]$ and the system has no energy at $n = 0$, the output $y[n] = h[n]$ is called the impulse response of the system



## Example

- Consider the DT system described by

$$y[n] + ay[n-1] = bx[n]$$

- Its impulse response can be found to be

$$h[n] = \begin{cases} (-a)^n b, & n = 0,1,2,\ldots \\ 0, & n = -1,-2,-3,\ldots \end{cases}$$

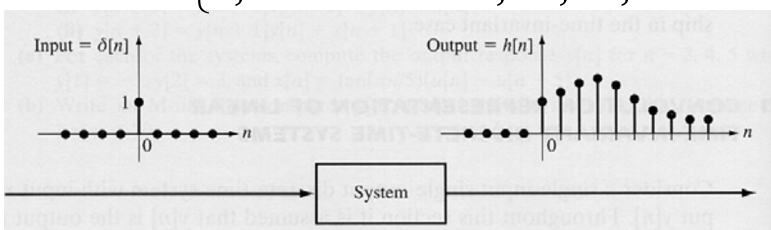



**Figure 3.1** Generation of the unit-pulse response.

## Representing Signals in Terms of Shifted and Scaled Impulses

- Let $x[n]$ be an arbitrary input signal to a DT LTI system
- Suppose that $x[n] = 0$ for $n = -1,-2,\ldots$
- This signal can be represented as

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

$$= \sum_{i=0}^{\infty} x[i]\delta[n-i], \quad n = 0,1,2,\ldots$$

## Exploiting Time-Invariance and Linearity

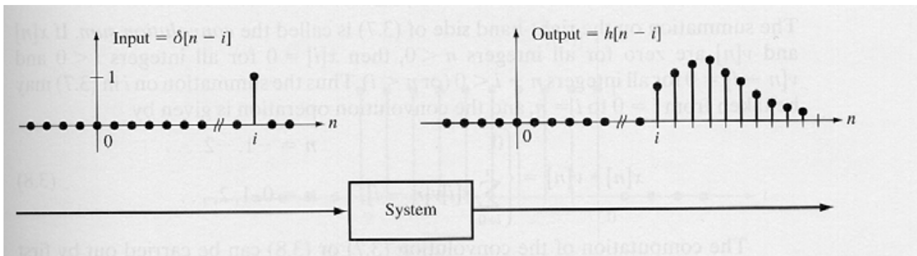



**Figure 3.2** The shifted unit-pulse response $h[n-i]$.

$$y[n] = \sum_{i=0}^{\infty} x[i]h[n-i], \quad n \geq 0$$

## The Convolution Sum

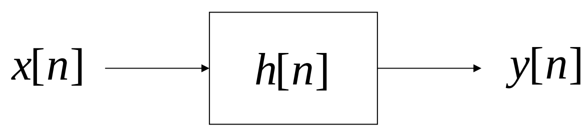
- This particular summation is called the convolution sum

$$y[n] = \underbrace{\sum_{i=0}^{\infty} x[i]h[n-i]}_{x[n]*h[n]}$$

- Equation $y[n] = x[n]*h[n]$ is called the *convolution representation of the system*
- Remark: a DT LTI system is completely described by its impulse response $h[n]$

## Block Diagram Representation of DT LTI Systems

- Since the impulse response $h[n]$ provides the complete description of a DT LTI system, we write




## The Convolution Sum for Noncausal Signals

- Suppose that we have two signals $x[n]$ and $v[n]$ that are not zero for negative times (noncausal signals)
- Then, their convolution is expressed by the two-sided series

$$y[n] = \sum_{i=-\infty}^{\infty} x[i]v[n-i]$$

## Example: Convolution of Two Rectangular Pulses

- Suppose that both $x[n]$ and $v[n]$ are equal to the rectangular pulse $p[n]$ (causal signal) depicted below

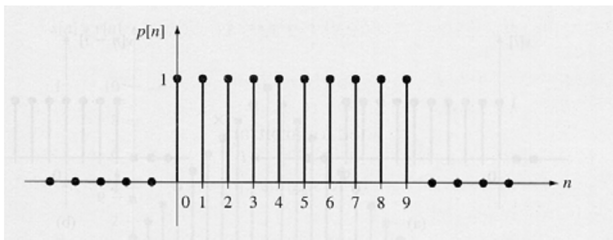



**Figure 3.3** The pulse $p[n]$.

## The Folded Pulse

- The signal $v[-i]$ is equal to the pulse $p[i]$ folded about the vertical axis

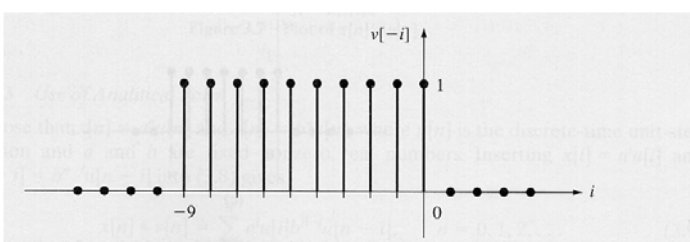



**Figure 3.4** The folded pulse.

## Sliding $v[n-i]$ over $x[i]$

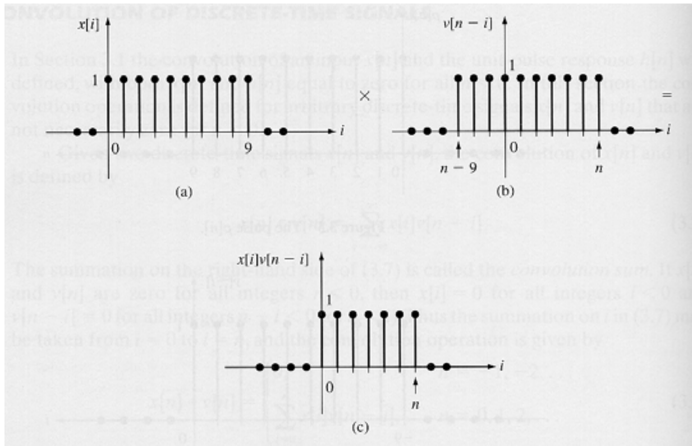



**Figure 3.5** Plots of (a) $x[i]$; (b) $v[n-i]$; (c) $x[i]v[n-i]$ for $0 \le n \le 9$.

## Sliding $v[n-i]$ over $x[i]$ - Cont'd

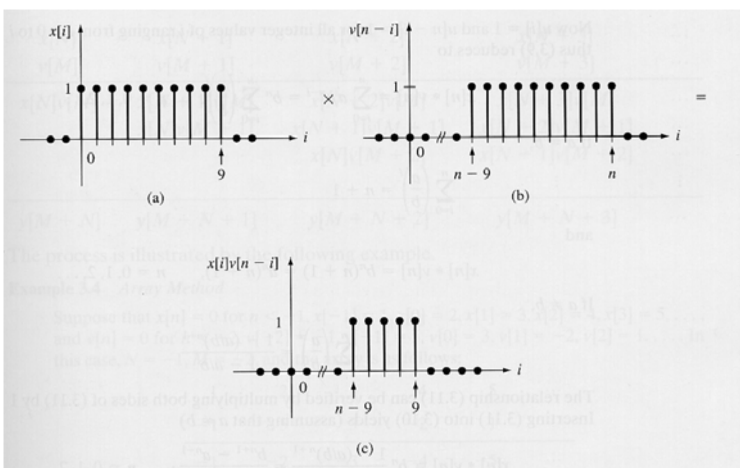



**Figure 3.6** Plots of (a) $x[i]$; (b) $v[n-i]$; (c) $x[i]v[n-i]$ for $10 \le n \le 18$.

## Plot of $x[n] * v[n]$

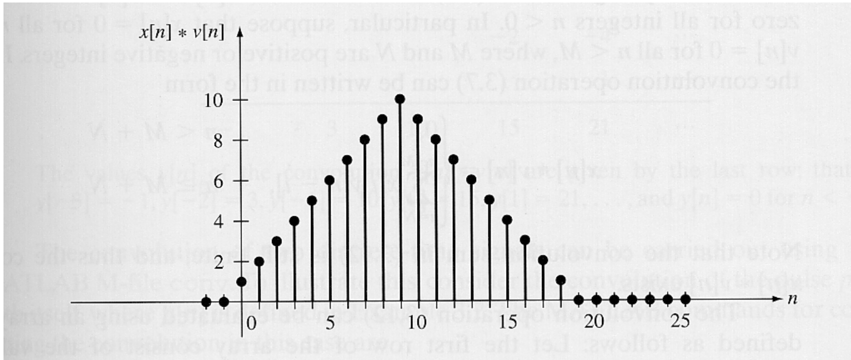



**Figure 3.7** Plot of $x[n] * v[n]$.

## Properties of the Convolution Sum

- Associativity

$$x[n] * (v[n] * w[n]) = (x[n] * v[n]) * w[n]$$

- Commutativity

$$x[n] * v[n] = v[n] * x[n]$$

- Distributivity w.r.t. addition

$$x[n] * (v[n] + w[n]) = x[n] * v[n] + x[n] * w[n]$$

## Properties of the Convolution Sum - Cont'd

- Shift property: define $\begin{cases} x_q[n] = x[n-q] \\ v_q[n] = v[n-q] \\ w[n] = x[n] * v[n] \end{cases}$

  then

$$w[n-q] = x_q[n] * v[n] = x[n] * v_q[n]$$

- Convolution with the unit impulse

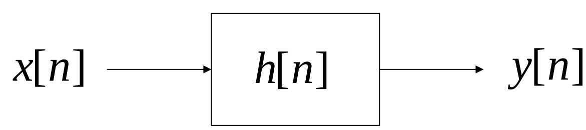
$$x[n] * \delta[n] = x[n]$$

- Convolution with the shifted unit impulse

$$x[n] * \delta_q[n] = x[n-q]$$

## Example: Computing Convolution with *Matlab*

- Consider the DT LTI system




- impulse response:

$$h[n] = \sin(0.5n), \quad n \geq 0$$

- input signal:

$$x[n] = \sin(0.2n), \quad n \geq 0$$

## Example: Computing Convolution with *Matlab* – Cont'd

$$h[n] = \sin(0.5n), \quad n \geq 0$$

$$x[n] = \sin(0.2n), \quad n \geq 0$$

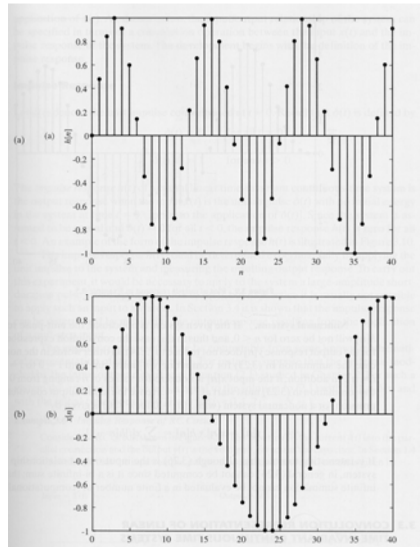


Figure 3.8 Plots of (a) h[n] and (b) x[n] in Example 3.5.

## Example: Computing Convolution with *Matlab* – Cont'd

- Suppose we want to compute $y[n]$ for $n = 0,1,\ldots,40$
- *Matlab* code:

```
n=0:40;
x=sin(0.2*n);
h=sin(0.5*n);
y=conv(x,h);
stem(n,y(1:length(n)))
```

## Example: Computing Convolution with *Matlab* – Cont'd

$$y[n] = x[n] * h[n]$$

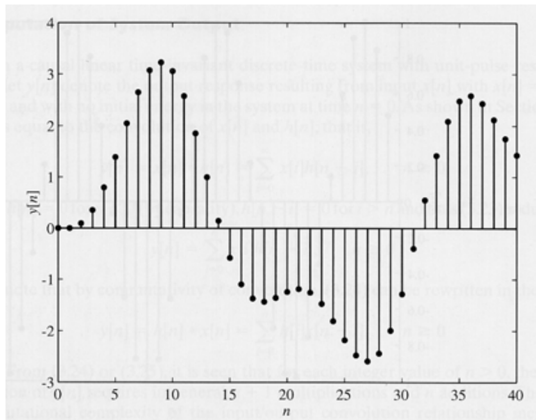
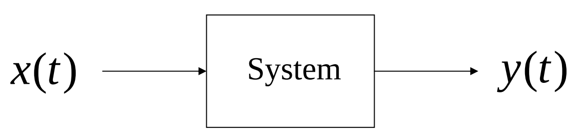


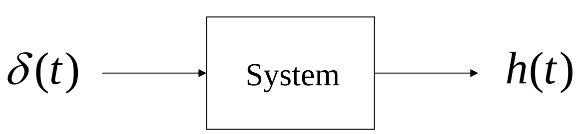

**Figure 3.9** Plot of output response in Example 3.5.

## CT Unit-Impulse Response

- Consider the CT SISO system:




- If the input signal is $x(t) = \delta(t)$ and the system has no energy at $t = 0^-$, the output $y(t) = h(t)$ is called the impulse response of the system



## Exploiting Time-Invariance

- Let $x[n]$ be an arbitrary input signal with $x(t) = 0, \text{ for } t < 0$
- Using the sifting property of $\delta(t)$, we may write

$$x(t) = \int_{0^-}^{\infty} x(\tau)\delta(t-\tau)d\tau, \quad t \geq 0$$

- Exploiting time-invariance, it is

$$\delta(t-\tau) \longrightarrow \boxed{\text{System}} \longrightarrow h(t-\tau)$$

## Exploiting Time-Invariance

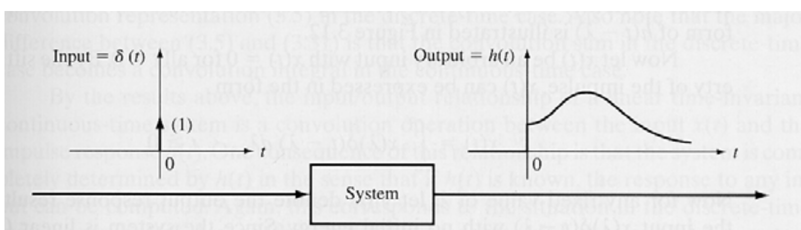


**Figure 3.10** Generation of the impulse response.

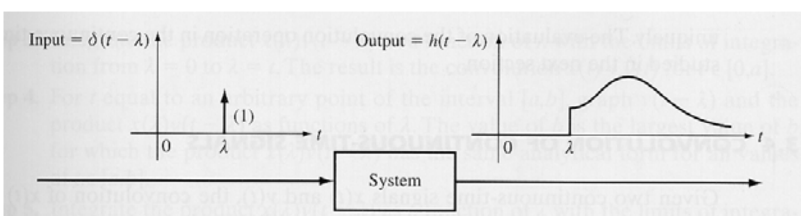


**Figure 3.12** Time-shifted impulse response $h(t-\lambda)$.

## Exploiting Linearity

- Exploiting linearity, it is

$$y(t) = \int_{0^-}^{\infty} x(\tau)h(t-\tau)d\tau, \quad t \geq 0$$

- If the integrand $x(\tau)h(t-\tau)$ does not contain an impulse located at $\tau = 0$, the lower limit of the integral can be taken to be 0,i.e.,

$$y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau, \quad t \geq 0$$

## The Convolution Integral

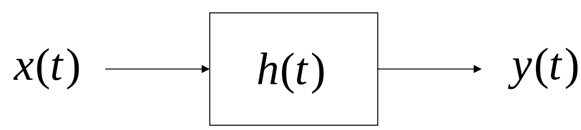
- This particular integration is called the convolution integral

$$y(t) = \underbrace{\int_{0}^{\infty} x(\tau)h(t-\tau)d\tau}_{x(t)*h(t)}, \quad t \geq 0$$

- Equation $y(t) = x(t) * h(t)$ is called the *convolution representation of the system*
- Remark: a CT LTI system is completely described by its impulse response $h(t)$

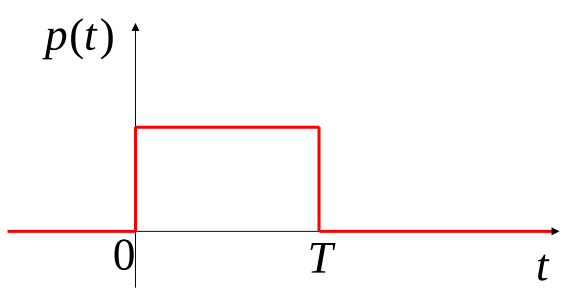
## Block Diagram Representation of CT LTI Systems

- Since the impulse response $h$(t) provides the complete description of a CT LTI system, we write




## Example: Analytical Computation of the Convolution Integral

- Suppose that $x(t) = h(t) = p(t)$, where $p(t)$ is the rectangular pulse depicted in figure



## Example – Cont'd

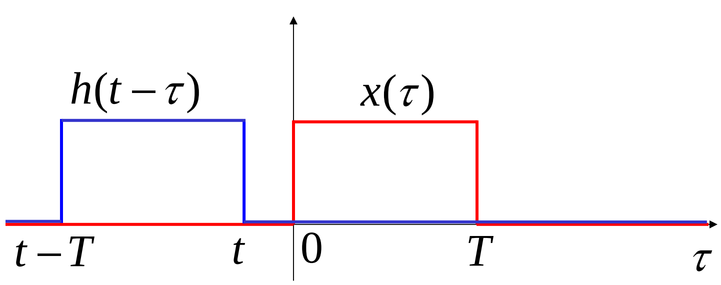
- In order to compute the convolution integral

$$y(t) = \int_0^{\infty} x(\tau)h(t-\tau)d\tau, \quad t \geq 0$$

we have to consider four cases:

## Example – Cont'd

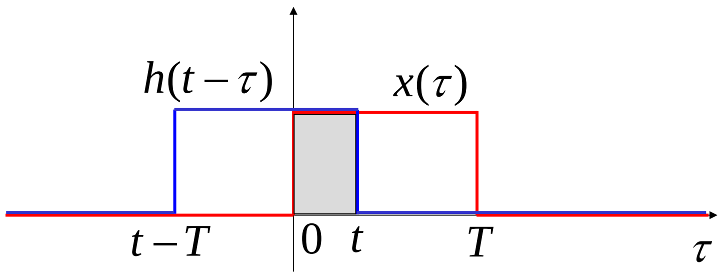
- Case 1: $t \leq 0$




$$y(t) = 0$$

## Example – Cont'd

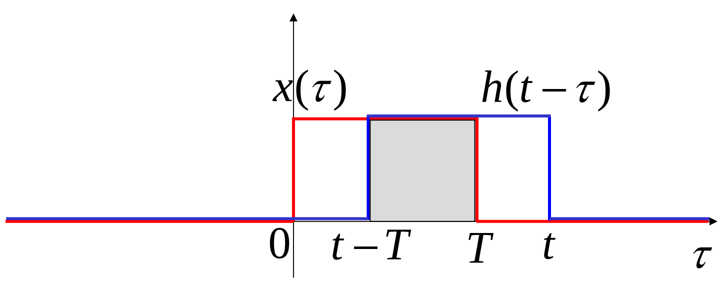
- Case 2: $0 \le t \le T$




$$y(t) = \int_0^t d\tau = t$$

## Example – Cont'd

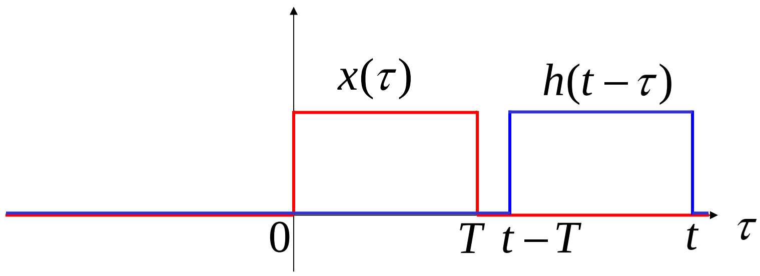
- Case 3: $0 \le t - T \le T \quad \rightarrow \quad T \le t \le 2T$




$$y(t) = \int_{t-T}^{T} d\tau = T - (t - T) = 2T - t$$

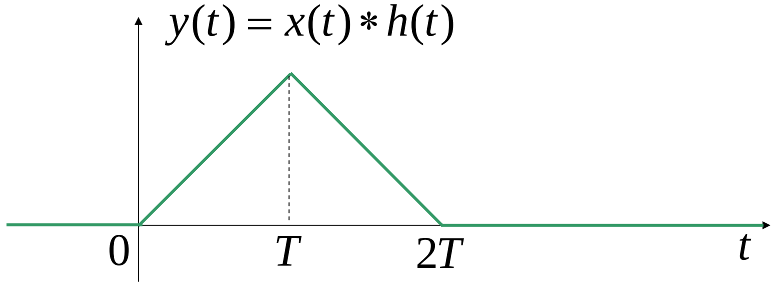
## Example – Cont'd

- Case 4: $T \le t - T \quad \rightarrow \quad 2T \le t$




$$y(t) = 0$$

## Example – Cont'd



## Properties of the Convolution Integral

- Associativity

$$x(t) * (v(t) * w(t)) = (x(t) * v(t)) * w(t)$$

- Commutativity

$$x(t) * v(t) = v(t) * x(t)$$

- Distributivity w.r.t. addition

$$x(t) * (v(t) + w(t)) = x(t) * v(t) + x(t) * w(t)$$

## Properties of the Convolution Integral - Cont'd

- Shift property: define

$$\begin{cases} x_q(t) = x(t-q) \\ v_q(t) = v(t-q) \\ w(t) = x(t) * v(t) \end{cases}$$

then

$$w(t-q) = x_q(t) * v(t) = x(t) * v_q(t)$$

- Convolution with the unit impulse

$$x(t) * \delta(t) = x(t)$$

- Convolution with the shifted unit impulse

$$x(t) * \delta_q(t) = x(t-q)$$

## Properties of the Convolution Integral - Cont'd

- Derivative property: if the signal $x(t)$ is differentiable, then it is

$$\frac{d}{dt}\left[x(t) * v(t)\right] = \frac{dx(t)}{dt} * v(t)$$

- If both $x(t)$ and $v(t)$ are differentiable, then it is also

$$\frac{d^2}{dt^2}\left[x(t) * v(t)\right] = \frac{dx(t)}{dt} * \frac{dv(t)}{dt}$$

## Properties of the Convolution Integral - Cont'd

- Integration property: define

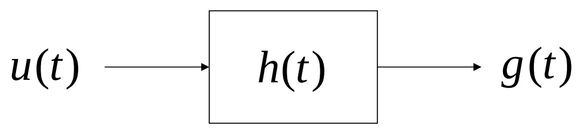
$$\begin{cases} x^{(-1)}(t) \doteq \int_{-\infty}^{t} x(\tau)d\tau \\ v^{(-1)}(t) \doteq \int_{-\infty}^{t} v(\tau)d\tau \end{cases}$$

then

$$(x * v)^{(-1)}(t) = x^{(-1)}(t) * v(t) = x(t) * v^{(-1)}(t)$$

## Representation of a CT LTI System in Terms of the Unit-Step Response

- Let $g(t)$ be the response of a system with impulse response $h(t)$ when $x(t) = u(t)$ with no initial energy at time $t = 0$, i.e.,




- Therefore, it is

$$g(t) = h(t) * u(t)$$

## Representation of a CT LTI System in Terms of the Unit-Step Response – Cont'd

- Differentiating both sides

$$\frac{dg(t)}{dt} = \frac{dh(t)}{dt} * u(t) = h(t) * \frac{du(t)}{dt}$$

- Recalling that

$$\frac{du(t)}{dt} = \delta(t) \quad \text{and} \quad h(t) = h(t) * \delta(t)$$

it is $\displaystyle \frac{dg(t)}{dt} = h(t)$ or $\displaystyle g(t) = \int_0^t h(\tau)d\tau$